GEOMETRIC EXTENSIONS AND THE 1/3 — 2/3 CONJECTURE

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San Francisco, California

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CERTIFICATION OF APPROVAL

I certify that I have read *GEOMETRIC EXTENSIONS AND THE 1/3 — 2/3 CONJECTURE* by Sam Sehayek and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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The 1/3—2/3 Conjecture is a famous open problem that deals with partially ordered sets, called posets. Understanding the linear extensions of a poset can unlock hidden structure with interesting applications and further questions. The conjecture says that in every finite poset that is not totally ordered there is a pair x and y, such that $x < y$ in 1/3 to 2/3 of all the linear extensions. Such pairs are called balanced pairs. We develop a geometric version of the conjecture, amenable to computational analysis by considering one dimensions projections of Euclidean realizations of the poset. We confirm the geometric 1/3—2/3 conjecture for certain classes of posets, for which the original 1/3—2/3 conjecture is currently out of reach. Ultimately, we derive quantitative estimates for the number of geometrically balanced pairs for the hom poset.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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Introduction

Finite partially ordered sets (known as posets) are classical objects in combinatorics. They have been studied from various perspectives, such as geometric, topological, number theoretical, etc. Posets are also used in a wide range of mathematical disciplines, ranging from abstract homotopy/homology theory to very concrete and difficult problems in graph theory [13]. Applications extend beyond mathematics, most notably in measuring relationships inscribed into large datasets that come from data or computer science. In those settings, the incomparable elements present an issue in analysis.

Trying to make sense of the poset structure in datasets gave rise to the famous open problem, known as the $1/3-2/3$ conjecture, which claims any poset admits a balanced pair, that is, a pair of incomparable elements that are ordered in a particular way in $1/3$ to $2/3$ of all of the linear extensions. Frequently, the proportion may be close to $1/2$, meaning there is an intrinsic property of posets that an incomparable pair might be switched to obtain a different linear extension.

The conjecture was originally pursued in the context of theoretical computer science [5], but is now a famous open problem in mathematics. The conjecture has remained unproven for decades, and progress for even very special classes of finite posets is coming along slowly. As an example, for the products of chains the conjecture was proved as recently as in 2017 [9]. A general approach that would unite all posets and lead to substantial progress on the conjecture has been missing. As a result, a proof of the general version of the conjecture seems out of reach.

In this paper we develop a geometric analog of the conjecture. One of the most fundamental facts about finite posets is that every poset admits a linear extension [14]. One way to see this is to use admissible 1-dimensional linear projections of an Euclidean (not necessarily planar) realization of the poset. We refer to these linear extensions as geometric extensions with respect to that realization. Not all linear extensions are geometric, even for small posets. But the family of geometric extensions of a given set is a structure by itself — the set of all linear (injective) projections of a particular Euclidean realization of a poset is a polytopal object. This polytope is subdivided by the equivalence classes — two projections are in the same class if they define the same linear extensions. Then, the volumes of the resulting sub-polytopes carry an important information on the poset. Namely, if for a particular incomparable pair x and y in the poset, the total volume of the cells leading to the extensions with $x < y$ is between $1/3$ and $2/3$ in proportion then the pair should be thought of as *geometrically balanced*. This approach makes it possible to apply polyhedral techniques to posets that have natural high-dimensional Euclidean embeddings. A notable example is the poset of all monotonic maps between two chains. Such posets have not been considered in the context of the original $1/3-2/3$ conjecture. Our work shows that they lead to rich polyhedral structures, and that geometrically balanced pairs can be detected in many cases. The technique we develop and apply allows us to get an estimate on the share of these pairs among the set of all incomparable monotonic maps, as measured by the appropriate volumes.

In Chapter 2, we introduce basic concepts for posets and their linear extensions and review the work done on the $1/3-2/3$ conjecture. We also make several general and non-difficult observations along the way. We spend several sections looking into particular classes of posets with the goal of understanding how they are linearly extended and whether there is evidence that the conjecture holds.

In Chapter 3, we introduce the poset structure on non-decreasing maps between chains, which is motivated by a categorial approach to posets. These posets are naturally modelled by order polytopes, and we use this formulation to look at their geometric extensions and study the geometric counterpart of the $1/3-2/3$ conjecture. In our investigation, we find ourselves dealing with high dimensional semi-algebraic regions in products of high-dimensional simplices. In lieu of a direct way to compute the volume of these regions, we approximate them by sampling the affine transformations of uniformly distributed random points obtained by the

Dirichlet distribution. As a result, we detect geometrically balanced pairs in certain infinite families of hom-posets. We actually obtain quantitative estimates of their occurrences using the volume computations. Finally, based on our analysis, we posit a conjecture that generalizes and strengthens the geometric analog of the $1/3-2/3$ conjecture for a concrete realization of the hom-poset between chains.

Chapter 2

Preliminaries

2.1 Finite Posets

The fundamental objects that are the subject of this thesis are finite posets. In this chapter, we will introduce the main objects of study and develop some of the tools employed. A good general reference can be found in [13].

Definition 2.1. A set P, along with an order relation $\preceq$, that is reflexive, transitive, and antisymmetric is called a *partially ordered set* or *poset*. Two elements $x, y \in P$ are *comparable*, denoted $x * y$, if $x \preceq y$ or $y \preceq x$. Otherwise, they are *incomparable*.

Some examples of posets are the integers ordered in the usual way, the divisors of 12 ordered by division (e.g. $2 \preceq 12$ because $2|12$), and the powerset of $\{1, 2, 3, 4\}$ ordered by inclusion (e.g., $\{2\} \preceq \{1, 2, 3\}$ because $\{2\} \subseteq \{1, 2, 3\}$, but $\{1, 2\}$ and $\{1, 4\}$ are incomparable). One way to deal with incomparable elements is to introduce a new relation that remembers all of the relations in the original poset. That new relation is an *extension* of P. The formal definition is as follows.

Definition 2.2. An *extension* or *refinement* of $(P, \preceq)$ is a new relation $\preceq'$ on P such that $a \preceq b$ implies $a \preceq' b$. A *linear extension* of P is an extension of P with no incomparable elements. In other words, it is a total order on P that is compatible with the partial order.

In general, linear extensions are not unique. That is, we can linearly extend a poset in many different ways. A simple example is the poset $\{1, 2, 3, 4, 5, a\}$ where the order relation is the usual order on the integers with a incomparable to the other elements. One linear extension is $(a, 1, 2, 3, 4, 5)$, but we also have $(1, a, 2, 3, 4, 5)$ and $(1, 2, a, 3, 4, 5)$, etc. This poset admits six linear extensions. It is clear in this example that a might be ordered before or after any of the other elements. A similar fact, proven by Szpilrajn, is true for all posets[14].

Lemma 2.1 (Szpilarajn 1930)**.** *If p and q are incomparable elements in a poset $(P, \preceq)$, then there is an extension of P with $p \preceq' q$.*

Proof. We introduce the relation $\preceq'$ on P where $x \preceq' y$ if and only if $x \preceq y$ or $x \preceq p$ and $q \preceq y$. This is an extension of $\preceq$ with $p \preceq' q$. □

In the rest of Szpilrajn's paper, he uses the axiom of choice in the form of Zorn's lemma to prove existence of linear extensions for any poset. We will present an alternate proof using geometry that acted as an inspiration for much of the work in this text. First, we need to develop visualization techniques for posets.

Definition 2.3. A *cover graph* for a poset is a drawing of $(P, \preceq)$ as a directed graph with vertices from P such that $x, y \in P$ are connected by a directed path if and only if $x * y$. In addition, there is an edge between comparable elements x and y if and only if there is no element between x and y (i.e., there is no $z \in P$ such that $x \prec z \prec y$). The edges represent a *cover relation.* A *Hasse diagram* of P is a cover graph with an implied upward orientation. That is, elements that are greater than another appear above it.

It is important to note that the definition above can be brought to bare for any poset.

Proposition 2.2. *Any poset has a Hasse diagram.*

Proof. We build the Hasse diagram algorithmically. Begin by placing elements that are smaller than any comparable element, i.e., $x \in P$ such that for all $y \in P$ with $x * y$ we have $x \preceq y$. Then, on the next level up, repeat the process with the rest of the elements. Make the necessary connections with the first line, and repeat. □

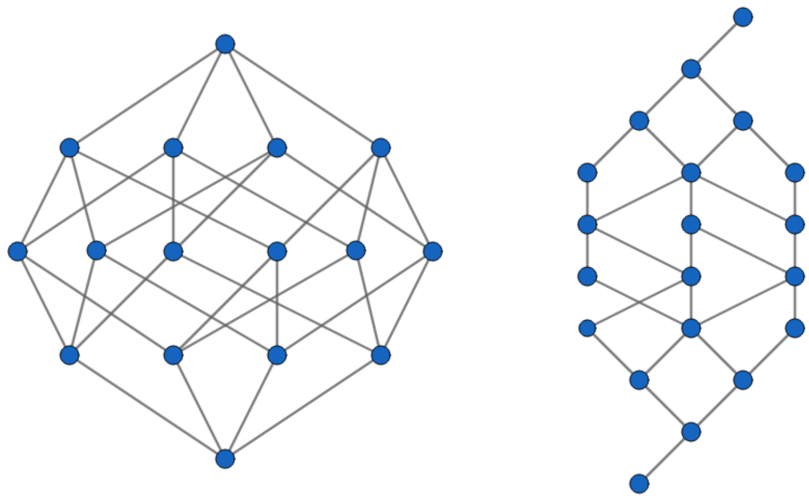

Figure 2.1: Hasse diagrams of the powerset of $\{1, 2, 3, 4\}$ ordered by inclusion and $\text{Hom}(C_3, C_4)$.

At times an embedding of the poset into higher dimensional space gives some insight. Hasse diagrams of a poset imply the existence of many such embeddings which will be used in Chapter 3.

Definition 2.4. A *Euclidean realization* of a finite poset $(P, \preceq)$ is an injective map

$$\theta : P \to \mathbb{R}^n \text{ for some } n.$$

It is worth noting that a Hasse diagram for a poset is a Euclidean realization into $\mathbb{R}^2$ with a few extra conditions. Now we can prove the existence of linear extensions geometrically.

Theorem 2.3. *Any finite poset P admits a linear extension.*

Proof. Consider the Hasse diagram as a Euclidean realization in $\mathbb{R}^2$ drawn such that greater elements appear directly above or right justified above smaller elements. Any two vertices have some slope between them. Since P is finite, there are only finitely many such slopes. If we take any line with positive slope not equal to any of the slopes between vertices, we can orthogonally project the Hasse diagram onto that line. The result is a Hasse diagram of a linear extension of P. Indeed, if $x \prec y$, then the image of these points on our line will hold that same relationship due to the line's positive slope. Since that slope was picked to be non parallel to any of the segments between two vertices, no pair of projections will land on the same point in the line. So, this is represents an extension of P that is totally ordered, so it is a linear extension. □

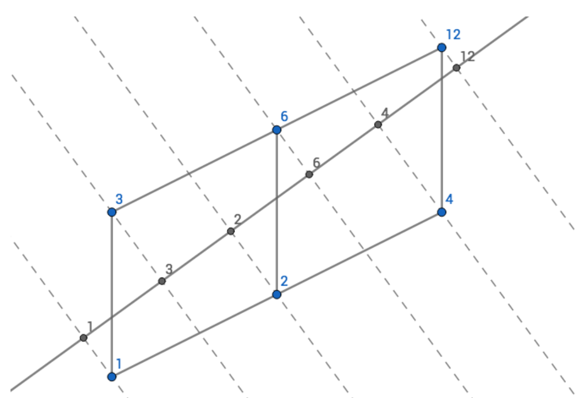

Figure 2.2: Illustration of linear extension from the Hasse diagram of the divisors of 12 ordered by division.

Lemma 2.1 already suggests that a poset that is not totally ordered always admits at least two linear extensions. There are frequently many more. Counting the total number of linear extensions can become a difficult exercise, so it is useful to understand what kind of structure a general linear extension may have. That brings us to the conjecture that is at the heart of our work, the $1/3 - 2/3$ Conjecture.

2.2 The $1/3 - 2/3$ Conjecture

Consider the set of all linear extensions of a poset $(P, \preceq)$, denoted $E(P)$. We can look at the subset of those that order a particular pair as $x < y$ (for convenience we use $<$ to mean x is ordered before y in a general linear extension). If x and y were comparable in the original poset, then all the extensions will retain that information. So, it is clear that $\mathbb{P}(x < y) \in \{0, 1\}$ in that case, where $\mathbb{P}(x < y)$ is the proportion of all linear extensions with $x < y$. If they are incomparable, however, this proportion is more interesting.

Lemma 2.4. *Let P be a poset. Then $0 < \mathbb{P}(x < y) < 1$ for any incomparable pair x, y.*

Proof. By Lemma 2.1, there is an extension of P with $x < y$ and another with $x > y$. □

The question becomes: how much better can the bounds on this proportion be improved? To answer this, the following was proposed.

Conjecture 2.1 (Kislitsyn 1968)**.** *In any finite poset, P, there exists a pair $x, y \in P$ such that*

$$\frac{1}{3} \leq \mathbb{P}_P(x < y) \leq \frac{2}{3}$$

where $\mathbb{P}_P(x < y)$ is the proportion of all linear extensions of P with $x < y$.

Such a pair x, y, if it exists, is called a *balanced pair.*

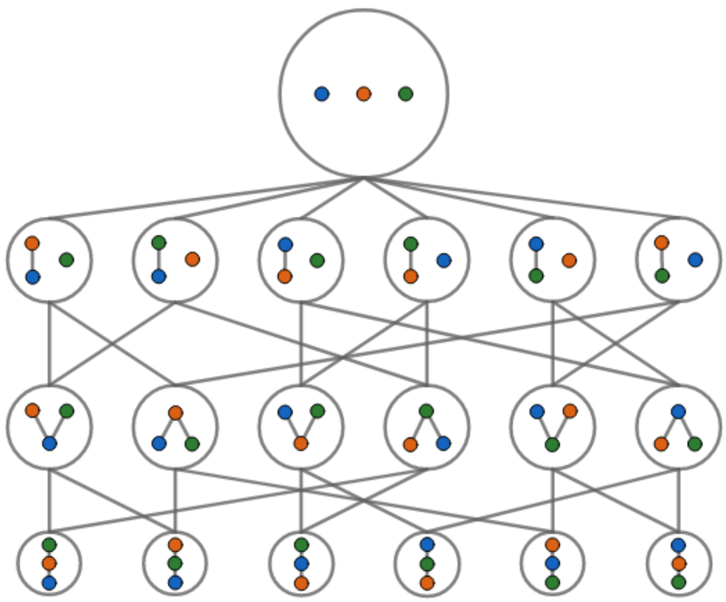

Figure 2.3: The poset of refinements of a 3 element antichain. Any pair is balanced.

The history of this conjecture begins when it was first proposed in 1968 by

Kislitsyn [7]. Alternate presentations were also proposed by Fredman in 1976 [5] and Linial in 1984 [8]. Though it is widely believed to be true, there is as of yet no proof. However, there has been significant interest in this problem. In 1995, it was proven that $\frac{5-\sqrt{5}}{10} \leq \mathbb{P}(x < y) \leq \frac{5+\sqrt{5}}{10}$ by Brightwell, Felsner, and Trotter [3]. If true, the 1/3 bound is the best possible (this can be seen by considering the 3 element poset with 1 relation — see the 2nd row from the top in Figure 2.3). There has also been progress in proving the conjecture in various classes of finite posets. Major classes have been resolved and are summarized in the following theorem.

Theorem 2.5. *Conjecture 2.1 holds for*

1. *Posets of width 2 [8];*
2. *N-free posets [15];*
3. *Posets with 11 or fewer elements [11];*
4. *6-thin posets[12];*
5. *Posets that admit a non-trivial automorphism [6];*
6. *Posets whose Hasse diagram is a tree [16];*
7. *The product of two chains [9].*

Now we turn our attention to certain basic classes of finite posets. We will investigate the conjecture by considering these classes and their linear extensions and whether there is proof the conjecture holds.

2.3 Disjoint Unions

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be disjoint finite posets. We will use $R := P \sqcup Q$ to denote the disjoint union of P and Q. In R, $a \leq b$ with $a, b \in R$ if $a, b \in P$ and $a \leq_P b$ or $a, b \in Q$ and $a \leq_Q b$. Any pair x, y with $x \in P$ and $y \in Q$ are incomparable.

The linear extensions of R are obtained by linearly ordering P and Q independently, then relating the elements of P to those of Q. Thus, the problem of counting linear extensions of R boils down to how many ways to arrange $|P|$ elements in $|P| + |Q|$ positions multiplied by the product of possible extensions for P and for Q. If $|P| = m$ and $|Q| = n$ and $e(X)$ is the number of linear extensions of a poset X then,

$$e(R) = \binom{m+n}{m} e(P)e(Q).$$

The statement for whether the conjecture holds can be formulated as follows:

Proposition 2.6. *If either P or Q has a balanced pair, then $R := P \sqcup Q$ has a balanced pair.*

Proof. Suppose without loss of generality that P has a balanced pair x, y. Then

$$\frac{1}{3} \leq \mathbb{P}_P(x < y) \leq \frac{2}{3}.$$

Claim: x, y is a balanced pair for R.

The number of extensions of R with $x < y$ must be a shuffling of P and Q in which $x <_P y$. So,

$$e(R \,|\, x < y) = \binom{m+n}{m} e(P \,|\, x < y) e(Q)$$

where $e(R \,|\, x < y)$ is the number of linear extensions of R with $x < y$. Since $\mathbb{P}_R(x < y) = \frac{e(R \,|\, x<y)}{e(R)}$ we have

$$\mathbb{P}_R(x < y) = \frac{\binom{m+n}{m} e(P \,|\, x < y) e(Q)}{\binom{m+n}{m} e(P) e(Q)} = \mathbb{P}_P(x < y),$$

so x, y is a balanced pair for R.

□

2.4 Branching Trees

The next class we will investigate are branching trees. We can completely characterize branching tree posets via their graphs.

Definition 2.5. A poset, P, is called a *tree* if its Hasse diagram is a connected acyclic graph, that is, a connected simple graph with no cycles. A *branching tree* is a tree with incomparable elements, i.e., it is not a total order.

Definition 2.6. A *forest* is a acyclic graph, not necessarily connected. A *branching forest* has incomparable elements on some connected component.

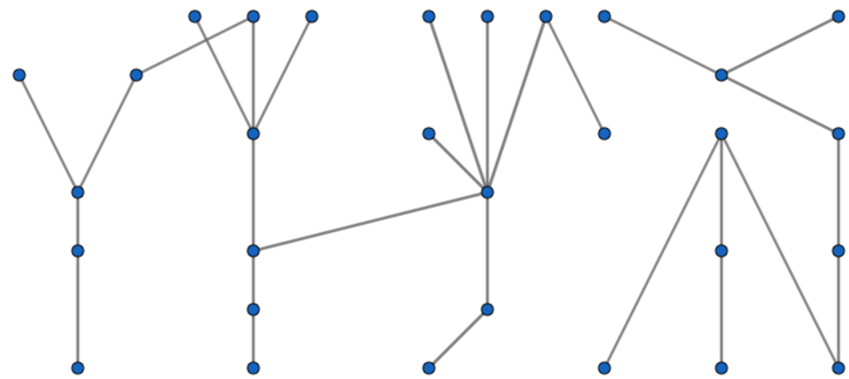

Figure 2.4: A Hasse diagram of a branching forest.

We want an idea for the size of $e(P)$, i.e., how many linear extensions are admitted on some branching tree. For simplicity, we consider branching trees with length 1 (only one cover relation away from the main chain of the poset). Our setup for a tree with m branches, is as follows. Consider the family of posets P with $m+n$ elements such that $a_1 < a_2 < \cdots < a_n$ and $a_{i_1} * b_1, a_{i_2} * b_2, \ldots, a_{i_m} * b_m$ and these are the only relations on P except for those that follow by transitivity.

2.4.1 All Maximal or All Minimal Branches

In this subsection, we will only focus on the simpler case when all the $*$ are the same relation, either all $<$ or all $>$. Since we are assuming the branches are length 1, every b_j is a maximal element and therefore *maximal branch* if the relation is $<$ and a *minimal branch* if the relation is $>$. We begin with the simplest case with only 1 maximal/minimal branch and build up in complexity.

1. $m = 1$: If $a_1 < \cdots < a_n$ and $a_{i_1} < b_1$ with $i_1 \in \{1, \ldots, n\}$ our poset P can be linearly ordered by inserting b_1 above a_{i_1} in any of the $(n - i_1 + 1)$ possibilities. If instead $a_{i_1} > b_1$ then there are exactly i_1 linear orderings obtained by inserting b below a_{i_1}.

2. $m = 2$: We have b_1, b_2 with $a_{i_1} < b_1$ and $a_{i_2} < b_2$ with $i_1 \leq i_2 \in \{1, \ldots, n\}$. Here, by first extending b_2 as in the $m = 1$ case and then by multiplying that by the number of choices for position of b_1 above a_{i_1}, we obtain the formula

$$e(P) = (n - i_2 + 1)(n - i_1 + 2).$$

   This captures the possible relationships between b_1 and b_2 in the linear order. If instead $a_{i_1} > b_1$ and $a_{i_2} > b_2$ then we now start the process of extending with the lower of the two so the formula becomes

$$e(P) = i_1(i_2 + 1).$$

3. $m = k$: If all the k branches are maximal with $i_1 \leq i_2 \leq \cdots \leq i_k \in \{1, \ldots, n\}$ then

$$e(P) = (n - i_k + 1)(n - m_{k-1} + 2) \cdots (n - i_1 + k).$$

On the other hand if all the k branches are minimal then

$$e(P) = (i_1)(i_2 + 1) \cdots (i_k + k - 1).$$

2.4.2 Maximal or Minimal Branches

By loosening the requirement that all the branch comparablities are the same relation, more care must be taken in enumerating the possible linear extensions of P.

Again consider the case where $m = 2$ but now $a_{i_1} < b_1$ and $a_{i_2} > b_2$. We can further divide this into 2 cases: if $i_1 \geq i_2$,

$$e(P) = (i_2)(n - i_1 + 1)$$

since in this case the extension of one does not really interact with the other. However, the other case $i_1 < i_2$ is more complicated. In this case,

$$e(P) = (i_2)(n - i_2 + 1) + (i_2 - i_1)(i_2 + 1).$$

This comes out of the problem that taking each step one at a time may give an undercount in the cases when the 1st step of extending contributes to how many nodes the 2nd step will choose from. So this closed formula can be read as "either b_1 is placed above a_{i_2} so there are i_2 possible placements of b_2 or b_1 is placed between

a_{i_1} and a_{i_2} and so there are $i_2 + 1$ possible placements of b_2."

The motivating generalization for the algorithm in the general case is when $b_1 > a_{i_1}$ but $b_2 < a_{i_2}, \ldots, b_m < a_{i_m}$ (Without loss of generality arrange such that $i_2 \leq i_3 \leq \cdots \leq i_m \in \{1, \ldots, n\}$). Here,

$$e(P) = \sum_{j=1}^{m} e(P \,|\, a_{i_j} < b_1 < a_{i_{1+j}}).$$

Computing these are careful applications of (3) from the previous subsection taking note of how the indices on a change once b_1 is integrated into the tree.

In lieu of a closed form counting formula for completely general length 1 branching trees, there is an algorithm for counting them that comes out of the above. The idea is that at each step we reduce the number of maximal branches by 1 until we are in the setting of (3) from the previous section then multiply and add the nested sum. The method would become clear after a few simple examples but is difficult to rigorously describe. The problem of counting the number of linear extensions for any tree (arbitrary branch length) is solved and can be computed in $O(n^2)$ time, where n is the total number of elements in the tree [1].

We already know from Theorem 2.5 that every poset whose Hasse diagram is a branching tree satisfies the conjecture. This result can also be extended to posets whose cover graphs are forests.

Corollary 2.7. *Any poset whose cover graph is a branching forest satisfies the* $1/3 - 2/3$ *Conjecture.*

Proof. A poset whose cover graph is a branching forest is a disjoint union of posets whose Hasse diagrams are branching trees. From Proposition 2.6 it follows that P has a balanced pair. □

2.5 Cartesian Products

The cartesian product of two posets P and Q is a new poset $R := P \times Q$ where $(x_1, y_1) \leq_R (x_2, y_2)$ if and only if $x_1 \leq_P x_2$ and $y_1 \leq_Q y_2$.

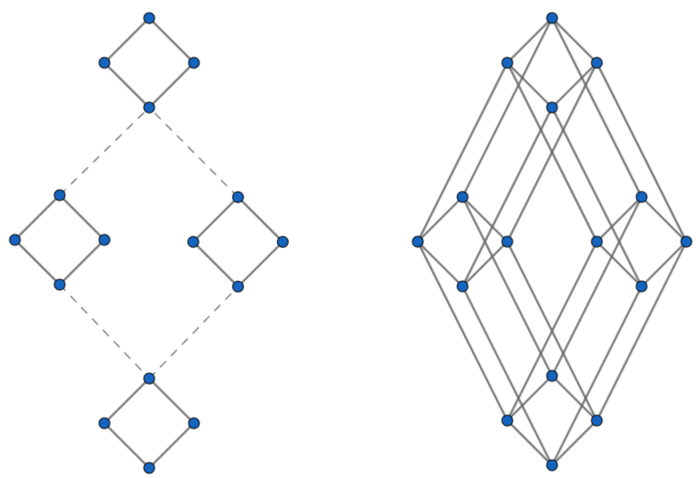

Figure 2.5: Constructing a Hasse diagram for the product of the diamond poset with itself.

We restrict our view to the case when P and Q are linearly ordered, i.e., when $P \times Q$ is the product of two chains. Even if P and Q are linearly ordered, the product poset is still only partially ordered.

If $|Q| = n = 2$ the total number of linear extensions for $P \times Q$ is encoded by the Catalan numbers; if $|P| = m$ then $e(P \times Q)$ is the mth Catalan number [13]. By Theorem 2.5, this poset satisfies the $1/3 - 2/3$ conjecture since it has a width of 2 [8]. The other cases (for other n) were also covered in Theorem 2.5.

If P and Q are not linearly ordered, there are fewer known results. But, given certain restrictions, there is something to be said.

Proposition 2.8. *Suppose P has an element incomparable to the rest of P and suppose Q has a balanced pair, then $R := P \times Q$ satisfies the $1/3 - 2/3$ Conjecture.*

Proof. Let $u \in P$ be incomparable to everything else in P. In the cross product, all the pairs of the form (u, q) for $q \in Q$ will be incomparable to everything else in $P \times Q$. The component (u, Q) is therefore part of a disjoint union with the rest. Since Q has a balanced pair, we apply Proposition 2.6. □

Chapter 3

Order Preserving Maps

Given two finite posets $(P, \leq_P)$ and $(Q, \leq_Q)$ we can look at the poset obtained by considering all order preserving maps from P to Q.

Definition 3.1. A map $f : P \to Q$ is called *order preserving* if for $x \leq_P y$ we have $f(x) \leq_Q f(y)$.

Definition 3.2. A map $f : P \to Q$ is called *order reflecting* if for $f(x) \leq_Q f(y)$ we have $x \leq_P y$.

The poset obtained by collecting all order preserving maps from P to Q defines $R := \mathrm{Hom}(P, Q)$ where $f \leq_R g$ if and only if $f(x) \leq_Q g(x)$ for all $x \in P$.

Definition 3.3. Two posets P and Q are *isomorphic as posets*, denoted $P \simeq Q$, if there is a bijective, order preserving, and order reflecting map from P to Q.

Proposition 3.1. $\mathrm{Hom}(P \sqcup Q, R) \simeq \mathrm{Hom}(P, R) \times \mathrm{Hom}(Q, R)$

Proof. Consider the map

$$\varphi : \mathrm{Hom}(P \sqcup Q, R) \to \mathrm{Hom}(P, R) \times \mathrm{Hom}(Q, R)$$

$$f \mapsto (f|_P, f|_Q).$$

Since $f \in \mathrm{Hom}(P \sqcup Q)$ is an order preserving map from $P \sqcup Q$ to R,

$$x \leq_{P\sqcup Q} y \text{ implies } f(x) \leq_R f(y)$$

and

$$x \leq_{P\sqcup Q} y \text{ implies } x \leq_P y \text{ or } x \leq_Q y$$

so $f|_P$ and $f|_Q$ are also order preserving. Indeed, they belong to $\mathrm{Hom}(P, R)$ and $\mathrm{Hom}(Q, R)$, respectively. So, φ is well defined.

To see that φ is order preserving, consider the pair $f, g \in \mathrm{Hom}(P \sqcup Q, R)$. If

$$f \leq_{\mathrm{Hom}(P\sqcup Q,R)} g \text{ that means } f(x) \leq_R g(x) \ \forall x \in P \sqcup Q.$$

In particular,

$$f|_P(x) \leq_R g|_P(x) \text{ for all } x \in P \text{ and } f|_Q(x) \leq_R g|_Q(x) \text{ for all } x \in Q.$$

So, $\varphi(f) \leq_{\mathrm{Hom}(P,R)\times\mathrm{Hom}(Q,R)} \varphi(g)$. By similar logic going backwards, the inverse also preserves order.

If $g \neq h \in \mathrm{Hom}(P \sqcup Q)$ then it is not the case that $g|_P = h|_P$ and $g|_Q = h|_Q$. That means either $g|_P \neq h|_P$ or $g|_Q \neq h|_Q$ which implies $\varphi(g) \neq \varphi(h)$. So, φ is injective.

It is also surjective because if $(f, g) \in \mathrm{Hom}(P, R) \times \mathrm{Hom}(Q, R)$ we can define

$$h(x) = \begin{cases} f(x) & x \in P, \\ g(x) & x \in Q, \end{cases}$$

then $h \in \mathrm{Hom}(P \sqcup Q, R)$ and $\varphi(h) = (f, g)$. Therefore, φ is a bijection and

$$\mathrm{Hom}(P \sqcup Q, R) \cong \mathrm{Hom}(P, R) \times \mathrm{Hom}(Q, R)$$

as desired. □

Proposition 3.2. $\mathrm{Hom}(P, Q \times R) \simeq \mathrm{Hom}(P, Q) \times \mathrm{Hom}(P, R)$.

Proof. Consider the map

$$\varphi : \mathrm{Hom}(P, Q) \times \mathrm{Hom}(P, R) \to \mathrm{Hom}(P, Q \times R)$$

$$(f, g) \mapsto h$$

where $h(x) = (f(x), g(x))$ for all $x \in P$.

When $x <_P y$, we must have that $f(x) \leq_Q f(y)$ and $g(x) \leq_R g(y)$ which implies $(f(x), g(x)) \leq_{Q\times R} (f(y), g(y))$ so φ is well-defined.

Injectivity and surjectivity for φ are trivial so this is a bijection.

To see it preserves order, consider the pair h_1, h_2 in $\mathrm{Hom}(P, Q\times R)$. If $h_1 <_{\mathrm{Hom}(P,Q\times R)} h_2$ then $h_1(x) <_{Q\times R} h_2(x)\ \forall x \in P$. So, $(f_1(x), g_1(x)) < (f_2(x), g_2(x))$ which gives us that $\varphi(h_1) < \varphi(h_2)$ in $\mathrm{Hom}(P, Q) \times \mathrm{Hom}(P, R)$ for all $x \in P$.

So, we have a bijective, order preserving map, so

$$\mathrm{Hom}(P, Q \times R) \simeq \mathrm{Hom}(P, Q) \times \mathrm{Hom}(P, R)$$

as desired. □

Definition 3.4. An isomorphism from a poset to itself, $\varphi : P \to P$, is called an *automorphism* on $(P, \preceq)$. In other words φ is an *automorphism* on P if it is a bijective, order preserving, and order reflecting map from P to P.

When a poset admits a non-trivial automorphism, we have confirmation of the conjecture. The result and proof, attributed to Pouzet [9], is the following.

Theorem 3.3. *(Pouzet) If a poset $(P, \preceq)$ admits an nontrivial automorphism, then there exists a balanced pair in P.*

Proof. Suppose φ is an automorphism on P. Then we must have that $\mathbb{P}(x < y) = \mathbb{P}(\varphi(x) < \varphi(y))$. Now, suppose there does not exist any balanced pair in P, i.e., there is no choice of $x, y \in P$ with $\frac{1}{3} \leq \mathbb{P}(x < y) \leq \frac{2}{3}$. Now, define the relation $<<$

on P such that $x << y$ if $\mathbb{P}(x \leq y) > \frac{2}{3}$. Our assumptions imply that $<<$ linearly extends P. Since φ must respect $<<$, φ must be the identity. So, if φ is not the identity on P, then there must be a balanced pair. □

3.0.1 Order Preserving Maps Geometrically

Consider all the order preserving maps from a given poset to $\mathbb{R}_{\geq 0}$. The result, $\mathrm{Hom}(P, \mathbb{R}_{\geq 0}) := K_P$ is no longer finite, but has a natural geometric interpretation. It is a polyhedral cone in $\mathbb{R}^{|P|}$ [2].

Definition 3.5. The *order cone* of a finite poset $(P, \leq_P)$ is the polyhedral cone

$$K_P = \left\{ f \in \mathbb{R}^P \ : \begin{array}{ll} 0 \leq f(x) & x \in P \\ f(x) \leq f(y) & x \leq_P y \end{array} \right\}.$$

If we add some element $\hat{1}$ that is greater than all the elements in P and denote it $\hat{P}$, we can intersect $K_{\hat{P}}$ with the hyperplane $\{f \in R^{\hat{P}} \,|\, f(\hat{1}) = 1\}$ to obtain a polytope called the *order polytope* of P.

Definition 3.6. The *order polytope* of a finite poset P is the polytope

$$O_P = \left\{ f \in \mathbb{R}^P \ : \begin{array}{ll} 0 \leq f(x) \leq 1 & x \in P \\ f(x) \leq f(y) & x \leq_P y \end{array} \right\}.$$

In defining these geometric objects, it is useful to identify maps as tuples using the isomorphism from $\mathbb{R}^P$ to $\mathbb{R}^{|P|}$.

3.0.2 Hom(P, Q) on linearly ordered sets

Just like in the product poset, the $\mathrm{Hom}(P, Q)$ poset already has interesting structure when P and Q are linearly ordered.

Let C_j be the chain with j elements. With no loss, we can associate C_j to the first j positive integers with the usual order. Now, we consider the poset $\mathrm{Hom}(C_m, C_n)$. An initial question concerns the size of this poset. To count all the functions in $\mathrm{Hom}(C_m, C_n)$, we just need to recognize this as the counting problem of making m-multisets from n elements. So the size is

$$|\mathrm{Hom}(C_m, C_n)| = \binom{n+m-1}{m}.$$

These posets and their Hasse diagrams become quite complicated very quickly, and so does the number of linear extension. For example $\mathrm{Hom}(C_2, C_4)$ has ten elements and admits 12 extensions, but $\mathrm{Hom}(C_2, C_5)$ has 15 elements and admits over a hundred extensions.

One reason to look at $\mathrm{Hom}(C_m, C_n)$ is that it admits geometric extensions in the same way as the product of chains poset. The construction is quite similar, except the hom-poset intrinsically lives in higher dimensions; instead of projecting onto a line, we project onto a hyperplane.

3.1 Geometric Extensions

Let $G = (V, E)$ be a graph that is the Hasse diagram of a poset P. There is always a one dimensional projection of G that agrees with the partial order from P. This 1-dimensional projection is naturally totally ordered, and thus, it is a linear extension of P. As usual, this projection can be done in different ways giving rise to a number of similarly obtained linear extensions. More generally,

Definition 3.7. A *geometric extension* of a poset $(P, \preceq)$ with respect to a Euclidean realization $\theta : P \to \mathbb{R}^n$ is an affine projection

$$\pi : \mathbb{R}^n \to \mathbb{R}$$

that is injective on the image of θ and such that if $x \prec y$ then $(\pi\theta)x < (\pi\theta)y$.

Any poset admits a geometric extension (with respect to some θ). Whats more, a geometric extension by construction induces a linear extension of P.

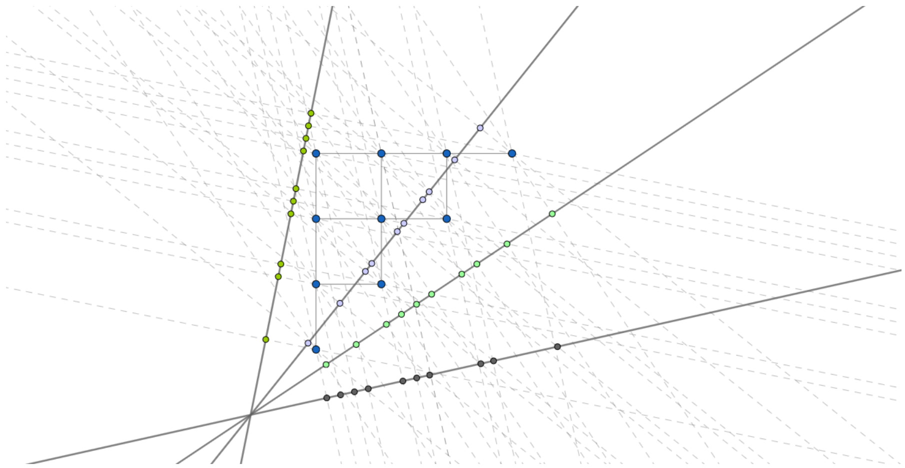

Figure 3.1: The standard realization of $\operatorname{Hom}(C_2, C_4)$ and its four geometric extensions.

Geometric extensions vary based on the diagram used. In many cases, the geometric extensions are only a subset of the possible linear extensions of a poset. The below figure is an example of a linear extension that is not a geometric extension by our definition.

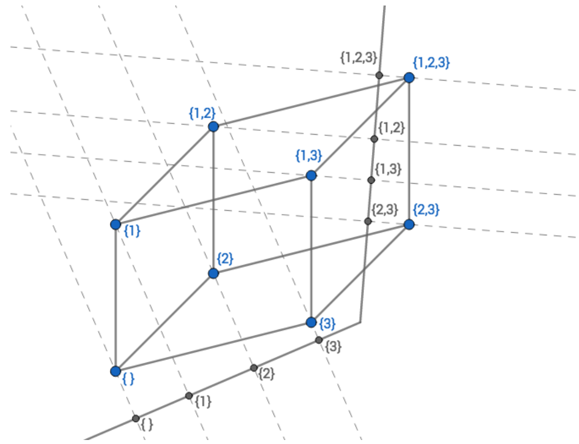

Figure 3.2: The bent line represents a linear extension of the subsets of $\{1, 2, 3\}$ ordered by inclusion. It is not a geometric extension with respect to this realization.

3.1.1 Geometric Extensions of Products of Chains

Given linearly ordered C_m, C_n, there is an association of the elements to consecutive natural numbers. This way $C_m \times C_n$ can be viewed as a subset of the upper right quadrant of $\mathbb{Z} \times \mathbb{Z}$. By connecting those lattice points by edges corresponding to the cover relations on the poset structure for $C_m \times C_n$, we obtain a quite natural Euclidean realization of $C_m \times C_n$ that will carry on throughout the rest of the text.

Any line through the origin with positive slope induces a projection of the $m \times n$ points onto it to give a new order.

As long as the slope of the norm is not parallel to the slope between any incomparable elements, this new order is a linear extension of $C_m \times C_n$. Any linear extension that can be obtained this way from this natural diagram will be called a *geometric extension* of $C_m \times C_n$.

Geometric extensions occur except on finitely many slopes though are generally non unique. Those slopes determine the total possible number of geometric extensions, $e_g(C_m \times C_n)$, by dividing the upper left quadrant into equivalence classes of geometric extensions. Elements remain in a particular order for all directions within each region, at the boundary some incomparable elements become indistinguishable, and on the other side, they swap order. Indeed, given the number μ of prohibited slopes

$$e_g(C_m \times C_n) = \mu + 1,$$

where μ is given by

$$\mu = (m-1)(n-1) - \left(\left\lfloor \frac{m-1}{p_1} \right\rfloor \left\lfloor \frac{n-1}{p_1} \right\rfloor + \cdots + \left\lfloor \frac{m-1}{p_k} \right\rfloor \left\lfloor \frac{n-1}{p_k} \right\rfloor \right)$$

for every prime $p_i \leq \min(m-1, n-1)$. If $m = n$, μ grows like the Farey sequence, giving this problem perhaps some interesting insight into a formulation of the Reiman Hypothesis (See section 3.1.2 for more information).

We can widen our inquiry to consider the poset structure on the product of more chains, i.e., $C_{a_1} \times C_{a_2} \times \cdots \times C_{a_n} = \prod^n C_{a_i}$ where C_{a_i} is the chain on a_i elements. There is a distinguished very natural Euclidean realization of these products.

Namely,

$$[a_1] \times \cdots \times [a_n]$$

is naturally mapped bijectively to the set of lattice points in the rectangular parallelepiped

$$[1, a_1] \times \cdots \times [1, a_n].$$

We call this realization of $\prod^n C_{a_i}$ the *standard Euclidean realization.* By abusing language, when we refer to a geometric extension of a product of chains, we will always mean a geometric extension with respect to the standard Euclidean realization.

3.1.2 Homogeneous Distribution of Weights

The one-dimensional projections of the product of two chains that give different geometric extensions are uniformly distributed, i.e., the prohibited slopes cut out generally equally sized equivalence classes.

More precisely, for $C_n \times C_n$, the formula for μ becomes

$$\mu = (n-1)^2 - \left(\left\lfloor \frac{n-1}{p_1} \right\rfloor^2 + \cdots + \left\lfloor \frac{n-1}{p_k} \right\rfloor^2 \right)$$

and we can recognize that this grows like the Farey sequence with n. The nth Farey sequence is the set of primitive integer directions within the triangle with vertices

$(0, 0)$, $(n, 0)$, and (n, n). It is known that the terms of Farey sequence rapidly become homogeneously distributed as n increases. In fact, Franel in 1924 [4] proved the best expected rate of this convergence,

$$\sum_{k=1}^{m_n} d_{n,k}^2 = O(n^r) \text{ for all } r > -1$$

for $d_{n,k} = a_{n,k} - k/m_n$ where $a_{n,k}$ is the kth term of the nth Farey sequence F_n, and m_n is the number of terms in F_n is equivalent to the Riemann Hypothesis. In the context of prohibited slopes, the Farey sequence represents a symmetric copy of this set. In the Figure 3.3, prohibited slopes are found to be every rational point of the form p/q in lowest terms with $p, q \leq n$ (for $n = 5$). The added condition that $p/q \leq 1$ makes the collection of slopes the nth Farey sequence.

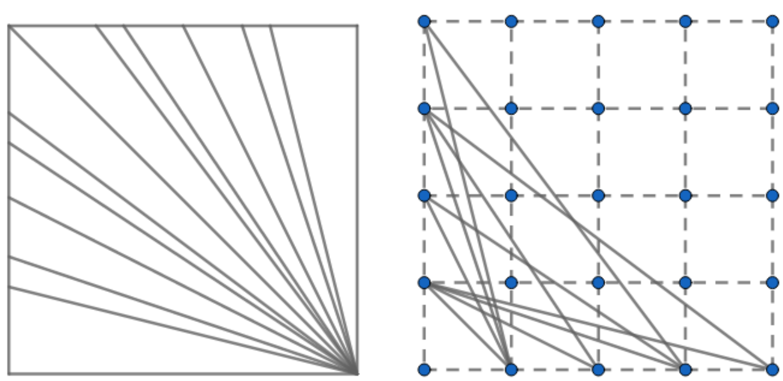

Figure 3.3: The prohibited slopes for $C_5 \times C_5$.

The segments on the right represent the prohibited projection directions and those on the left are (two copies of) a representation of a symmetric inverted Farey sequence.

We can expect the same homogeneous distribution phenomenon in high dimensions. In the next section, we will identify the set $\mathrm{Hom}(C_m, C_n)$ with the set of lattice points in a simplex embedded in the cube $[0, n]^m$ which is the higher dimensional analog of the square in Figure 3.3. Then the geometric extensions for this poset will be defined by hyperplanes not passing through any pair of incomparable lattice points. The high-dimensional analog of the homogeneity we have in dimension 2 is controlled by the family of cones cut out by the hyperplanes through the lines of these pairs.

We expect, in the spirit of Frenel's result, that the spans of these cones, as measured by the intersection with the unit sphere centered at the origin, become rapidly homogeneous as n goes to infinity.

Based on this expectation, computing the volume of a region in the space of directions amounts to computing the proportion of the geometric extensions resulting from the cones passing through this region. This framework leads to the estimates of the proportion of geometrically balanced pairs among all pairs. For more details, see Observations 3.1–3.7.

3.1.3 Geometric Extensions on $\mathrm{Hom}(C_m, C_n)$

Remark. $\mathrm{Hom}(C_m, C_n)$ is naturally a subposet of C_n^m so as we did with the product of chains, when we refer to a geometric extension of $\mathrm{Hom}(C_m, C_n)$, we always mean a geometric extension with respect to the Euclidean realization which is the restriction of the standard realization. When necessary, we will refer to this realization as ϑ.

Thus we can think of a map in $\mathrm{Hom}(C_m, C_n)$ as a tuple in $\mathbb{Z}^m$, i.e., for $f \in \mathrm{Hom}(C_m, C_n)$ we have $f = (x_1, x_2, \ldots, x_m)$ with $1 \leq x_1 \leq x_2 \leq \cdots \leq x_m \leq n$. This association makes clear that the standard realization as a diagram for this poset is contained as as subgraph to that of the product poset C_n^m (connecting lattice points with edges in $\mathbb{Z}^m$). Taking all such tuples, this defines a simplex in $\mathbb{Z}^m$. We also define $\Gamma_m \subset \mathbb{R}^m$ to be the simplex defined by $x_1 + \cdots + x_m = 1$ with $x_i \geq 0$ for each i.

Projecting along $w \in \Gamma_m$, we can define the relation $f \prec g$ if and only if $\sum w_i f(i) < \sum w_i g(i)$. For $f = (a_1, \ldots, a_m)$ and $g = (b_1, \ldots, b_m)$ this is $\sum a_i w_i < \sum b_i w_i$. For any choice of $w \in \Gamma_m$ this relation agrees with the partial order on $\mathrm{Hom}(C_m, C_n)$. Indeed, if $g(x) \geq f(x)$ for all $x \in P$, then $\sum w_i f(i) \leq \sum w_i g(i)$. Certain choices of the w's will give rise to linear extensions of $\mathrm{Hom}(C_m, C_n)$. Because of the uniform distribution of prohibited directions, the weights partition the simplex Γ_m into equivalence classes each of which gives rise to the same extension.

Problems arise when the weighted sum is not unique. Take, for example, $m = 3 < n$ $f(1) = f(2) = f(3) = 2$ and $g(1) = g(2) = 1$ but $g(3) = 4$. The point inside the simplex $w = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ gives $\sum w_i f(i) = \sum a_i g(i)$. So we should have that $f = g$, since that is not the case, this point w does not gives rise to a partial order, let alone a linear order (antisymmetry fails).

This example generalizes to $m < n$, then $w = (\frac{1}{m}, \ldots, \frac{1}{m})$ the point on the $m-1$ dimensional simplex Γ_m, does not extend $\mathrm{Hom}(C_m, C_n)$. It cannot be an extension because the similar choice of f and g as above would force antisymmetry to fail.There are other problematic points in Γ_m but still only finitely many and are negligible given the right point of view.

3.2 Geometrically Balanced Pairs

We are interested in pairs of maps, so by our construction pairs of points from a simplex, in which one is favored to the other in between $1/2 - \varepsilon$ and $1/2 + \varepsilon$ of geometric extensions with respect to our natural diagram. Pairs with this property will be called *geometrically ε-balanced pairs.*

For this we introduce the following. We consider the tuples in $\mathrm{Hom}(C_m, C_n)$. When n is arbitrarily large, scaling each component by n gives a number less than or equal to 1. In other words, $\mathrm{Hom}(C_m, C_n)$ naturally identifies with the set of lattice points in the n-dilate of the following geometric object, a simplex $\Delta_m := \{\mathbf{x} \in \mathbb{R}^m \,|\, 0 < x_1 \leq x_2 \leq \cdots \leq x_m \leq 1\}$. The closure of Δ_m is the order polytope of C_m.

Now define the map

$$f_m : \Delta_m \times \Delta_m \to [0, 1]$$

that sends $(a, b) \in \Delta_m \times \Delta_m \mapsto p \in [0, 1]$ where p is the proportion of weights from Γ_m that favor a to b.

Virtually all those weights give rise to a geometric linear extension. That means the output of our function gives the proportion of weights that place a before b.

For small m, the map f_m can be manually computed. The computations for $m = 2$ and $m = 3$ are as follows. When $m = 2$, we have that $\Delta_2 \times \Delta_2$ is a duo-prism.

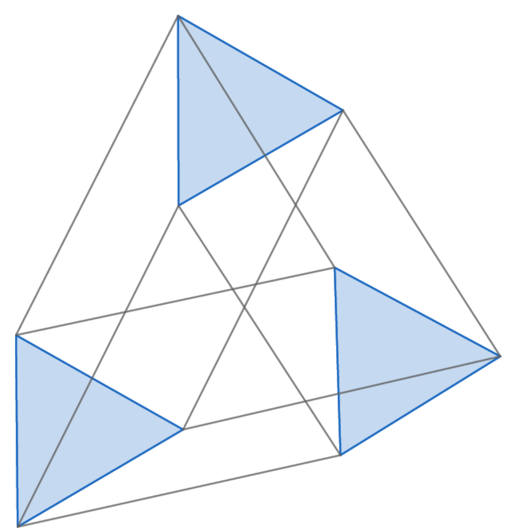

Figure 3.4: A projection of $\Delta_2 \times \Delta_2$ into 3-space.

Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$. We first consider

$$a_1 w_1 + a_2 w_2 < b_1 w_1 + b_2 w_2$$

$$0 < (b_1 - a_1) w_1 + (b_2 - a_2) w_2.$$

Finding the intersection with Γ_2 will give us the desired proportion. To find that intersection we substitute $w_2 = 1 - w_1$ and consider the following two cases:

If $b_2 > a_2$, then $w_1 < \frac{b_2 - a_2}{((b_2 - a_2) - (b_1 - a_1))}$, and translating to a proportion of Γ_2 gives

$$f_2(a, b) = \frac{b_2 - a_2}{((b_2 - a_2) - (b_1 - a_1))}.$$

If $b_2 < a_2$ then we have $w_1 > \frac{b_2 - a_2}{((b_2 - a_2) - (b_1 - a_1))}$ and so

$$f_2(a, b) = \frac{b_1 - a_1}{((b_1 - a_1) - (b_2 - a_2))}.$$

For $m = 3$, we have $\Delta_3 \times \Delta_3$ is a duoprism in 6 dimensions.

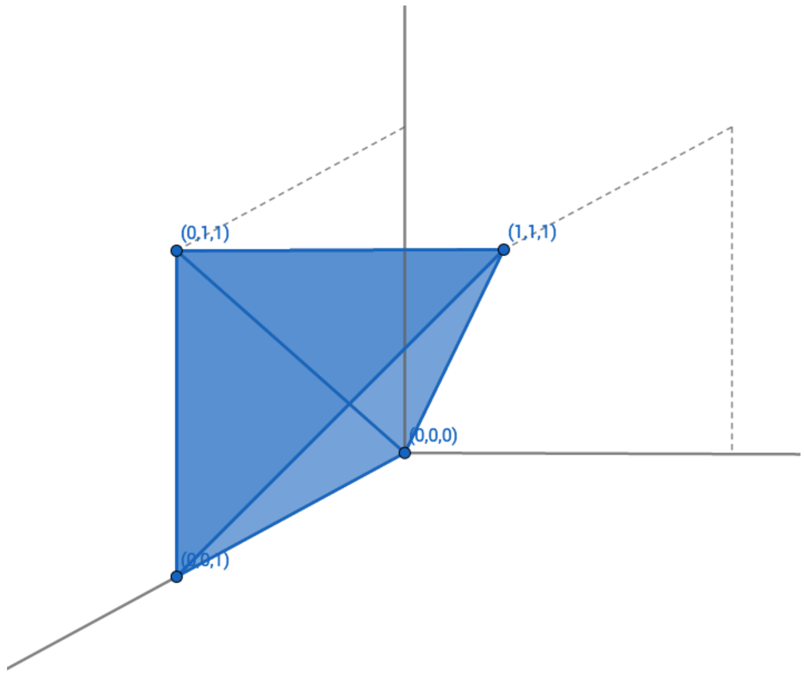

Figure 3.5: The simplex Δ_3 sitting in $\mathbb{R}^3$.

Let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$. To compute $f_3(a, b)$ we begin again by comparing the sums:

$$a_1 w_1 + a_2 w_2 + a_3 w_3 < b_1 w_1 + b_2 w_2 + b_3 w_3.$$

Combining like terms gives

$$(b_1 - a_1)w_1 + (b_2 - a_2)w_2 + (b_3 - a_3)w_3 > 0.$$

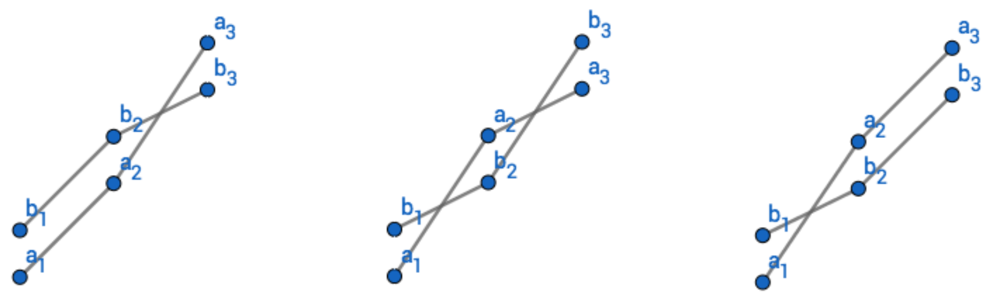

Figure 3.6: The three types of incomparable pairs in Δ_3.

Comparing the differences in the components of a and b, we can see from Figure 3.6 above that one of the signs of these differences is opposite to the other two. To get the correct volume of Γ_3, we must choose that difference. Say $(b_3 - a_3)$ is the difference with the different sign. (The computation in the other cases are analogous.) We get

$$-(b_3 - a_3) < ((b_1 - a_1) - (b_3 - a_3))w_1 + ((b_2 - a_2) - (b_3 - a_3))w_2.$$

This area is the convex hull of $\big(0, -\frac{b_3-a_3}{(b_2-a_2)-(b_3-a_3)}, \frac{b_2-a_2}{(b_2-a_2)-(b_3-a_3))}\big)$, $\big(-\frac{b_3-a_3}{(b_1-a_1)-(b_3-a_3)}, 0, \frac{b_1-a_1}{(b_2-a_2)-(b_3-a_3))}\big)$, and $(0,0,1) \in \mathbb{R}^3$ or its complement with respect to Γ_3 if $(b_3 - a_3) > 0$ or $(b_3 - a_3) < 0$, respectively.

Using the $2\pi/3$ radian angle, the area of the triangle can be computed to be

$$\frac{\sqrt{3}(b_3 - a_3)^2}{2((b_2 - a_2) - (b_3 - a_3))((b_1 - a_1) - (b_3 - a_3))}$$

This area as a proportion of the total is

$$\frac{(b_3 - a_3)^2}{((b_2 - a_2) - (b_3 - a_3))((b_1 - a_1) - (b_3 - a_3))}.$$

Based on the sign of $(b_3 - a_3)$ we get two cases:

If $(b_3 - a_3) > 0$ we have

$$f_3(a, b) = \frac{(b_3 - a_3)^2}{((b_2 - a_2) - (b_3 - a_3))((b_1 - a_1) - (b_3 - a_3))}.$$

If $(b_3 - a_3) < 0$ we have

$$f_3(a, b) = 1 - \frac{(b_3 - a_3)^2}{((b_2 - a_2) - (b_3 - a_3))((b_1 - a_1) - (b_3 - a_3))}.$$

The other cases gives rise to the same formula up to a permutation of indices. If $b_2 - a_2$ has the different sign,

$$f_3(a, b) = \frac{(b_2 - a_2)^2}{((b_3 - a_3) - (b_2 - a_2))((b_1 - a_1) - (b_2 - a_2))}.$$

If $b_1 - a_1$ has the different sign,

$$f_3(a, b) = \frac{(b_1 - a_1)^2}{((b_3 - a_3) - (b_1 - a_1))((b_2 - a_2) - (b_1 - a_1))}.$$

The choice of the numerator should be the difference with the different sign. This is guaranteed to exist without ambiguity unless $a_i = b_i$ (this can only happen with one i, otherwise they are comparable anyway), in which case, either choice is fine, keeping in mind the sign.

Once we go to $m > 3$, the cases become harder to enumerate. This is because the sections of Γ_m will not be as clearly delineated. For $m = 2$ and $m = 3$, the cases arose out of the necessity of one difference having a different sign than the others. When $m = 4$ or more, we no longer have that guarantee. There may not be some difference with a unique sign. The complication that comes out of those cases will be that the intersection with Γ_m will not cut out a corner of the simplex, but instead may cut out some different chunk.

That said, for all m, we will have certain cases where we will cut out one corner of the simplex Γ_m.

When there is one difference that has a different sign, then it is the perturbation of the weight on that component that decides whether the extension will favor one or the other. So, in those cases, we can more easily determine the cut out.

We follow the same logic as in the $m = 3$ case. Once we find the region of the edges that determine the corner, we multiply those proportions of the edge size and that is the proportion of the volume compared to that of Γ_m. What this means is that if there is one difference in components with a different sign then all the others, we have for $m = k$,

$$f_k(a, b) = \frac{(b_i - a_i)^{k-1}}{\prod_{j \neq i} \big((b_j - a_j) - (b_i - a_i)\big)}.$$

We note that this is not true for all the possible incomparable pairs. This formula comes into play when there is a difference component-wise that has a different sign to all the other differences. For Δ_2 and Δ_3, the absence of a unique difference with a different sign means the two points chosen are comparable. Indeed, here are the possibilities for Δ_3:

1. the signs are all 3 positive — in which case $b > a$;
2. the signs are all negative — so $b < a$,
3. They are all zero — so $b = a$;
4. Two are zero — so $b < a$ or $b > a$ depending on the sign of the third difference;
5. One is zero and the others all positive or negative — so $b > a$ or $b < a$ respectively;
6. If one is zero and the others are different, or if two are the same and the other different signs — then the pair is incomparable.

For higher dimensions, incomparable pairs may have more than one of each sign when looking at the differences. That said, the ones with a unique different sign constitute some nonzero number of the incomparabilities.

3.3 Preimage Volumes

Much of our motivation to compute f_m is that there is an interesting interpretation for its preimages. Originally, this preimage gives the proportion of weights that order a pair in a particular way. But that proportion asymptotically should be the proportion of geometric extensions themselves. Indeed, if our earlier claim is true, the preimage of the region $[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon]$ are the geometrically ε-balanced pairs of Δ_m. We would like to see the preimage of geometrically ε-balanced pairs in $\Delta_m \times \Delta_m$, at least for certain values of m when f_m can be readily computed. Methods for computing the volume directly is difficult. Afterall, the region in $\Delta_m \times \Delta_m$ cut out by $f_m^{-1}([\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon])$ is non-convex in general and defined by high-degree inequalities, it is a semi-algebraic set. Another difficulty is that they live in high dimensions. In order to efficiently approximate the volume, we turn to some statistical tools and employ a Monte Carlo sampling to find the proportion of geometrically ε-balanced pairs in $\Delta_m \times \Delta_m$. This technique can be described as taking random points in our set and testing whether or not they lay in the region we are interested in. Using a high number of randomly generated points, we can get close to the true probability that a point lays in the region, which is a direct analog to the volume we want. To generate our random points uniformly, we use the Dirichlet distribution with parameter $\alpha = (1, \ldots, 1)$ of the right size. That generates a point in Γ_{m+1} which we can affinely transform to Δ_m. This transformation is as follows:

$$T : \Gamma_{m+1} \to \Delta_m$$

that sends

$$(x_1, \ldots, x_{m+1}) \mapsto \Big(x_1, x_1 + x_2, \ldots, \sum_{i=1}^{m} x_i\Big).$$

We collect the total number of hits and display the result for varying values of ε. For $\Delta_2 \times \Delta_2$, we have already concluded a linear relationship, and we see this in Figure 3.7 as well as a successful test case.

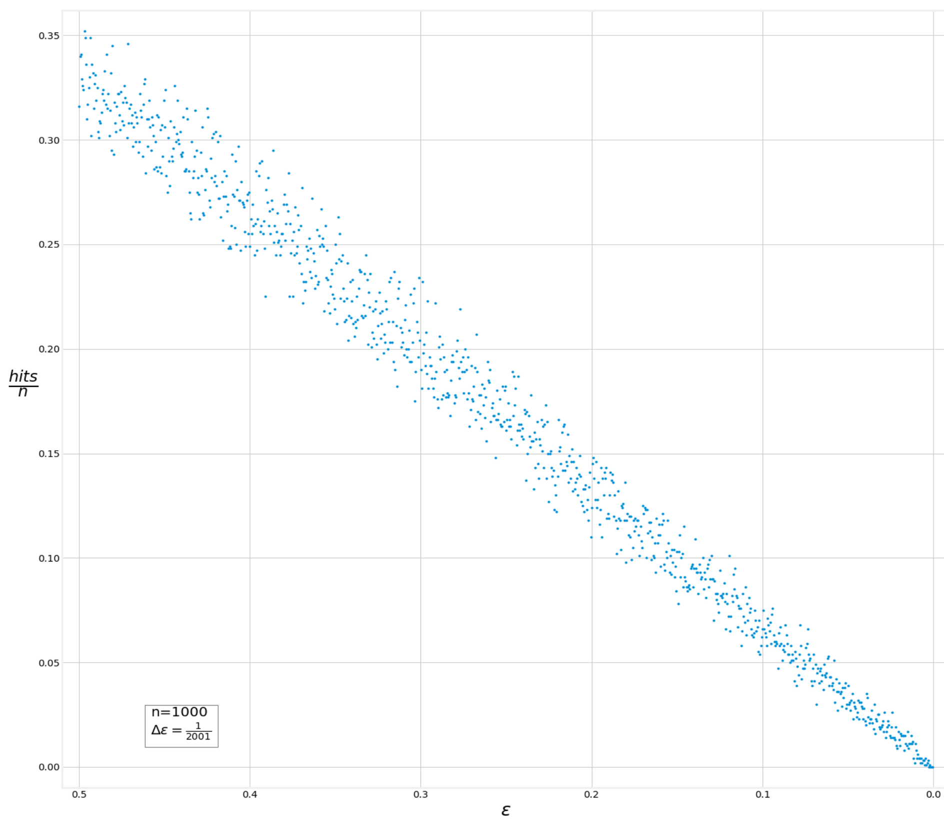

Figure 3.7: The proportion of geometrically ε-balanced pairs in $\Delta_2 \times \Delta_2$ as ε changes.

Remark. In Figures 3.7—3.11, the x-axis takes on values for ε between 0 and 1/2, and the graph returns $f^{-1}([\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon])$ as a proportion of the total volume. Each data point on the graph represents this process for each of 2000 choices of ε, distributed evenly in $[0, .5]$.

In Figure 3.8, we can see this method in use for f_3^{-1} with more intersting results.

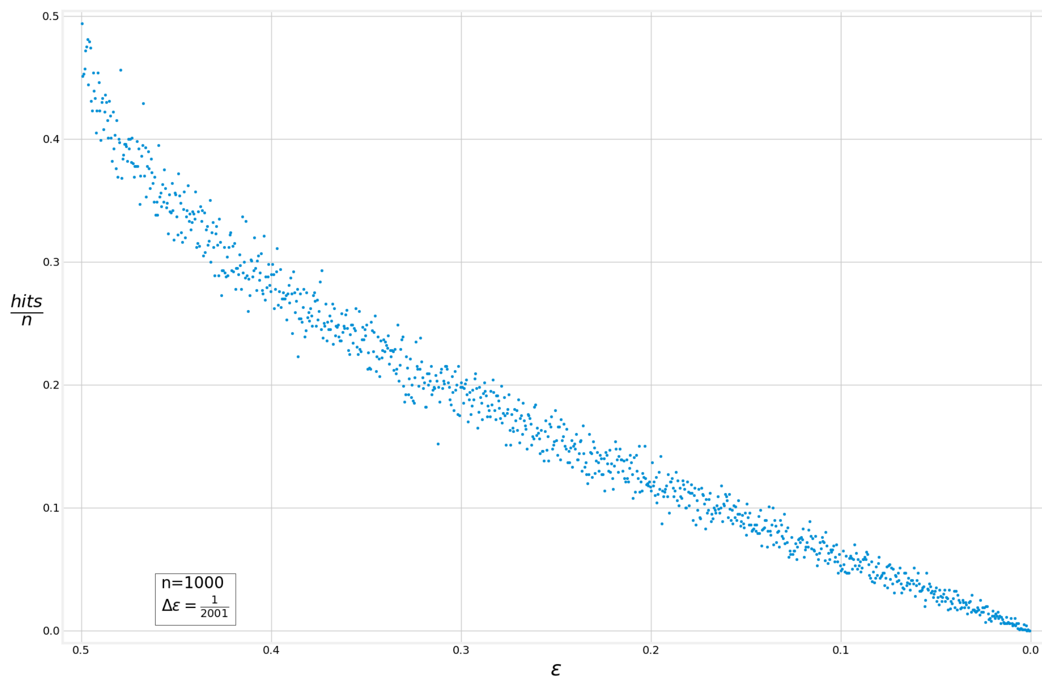

Figure 3.8: The proportion of geometrically balanced pairs in $\Delta_3 \times \Delta_3$ as ε changes.

Observation 3.1. The first striking thing about Figure 3.8 is that it is not the zero function. There is a large number in proportion of these ε-geometrically balanced pairs in $\Delta_3 \times \Delta_3$.

Observation 3.2. Another interesting component is when ε is $1/6$ because those are the geometrically balanced pairs from the geometric version of the conjecture.

At that value, we see that around a tenth of the pairs in $\Delta_3 \times \Delta_3$ are geometrically balanced with the $1/3 - 2/3$ bound.

We can push this model into higher dimensions. The only caveat with this program is that the formula for f_m is defined for the case when there is a unique difference. That is exhaustive when m is 2 or 3, but not for greater values of m. Therefore, the interpretation of our models going forward should be as a lower bound of the proportion of geometrically ε-balanced pairs. The graphs for $\Delta_4, \Delta_5, \Delta_6$ are below. The accompanying code uses the same inline plotting system so the included code is what is different in each dimension. The biggest change is recording an automatic miss whenever there is no difference with a different sign in a pair of randomly chosen points.

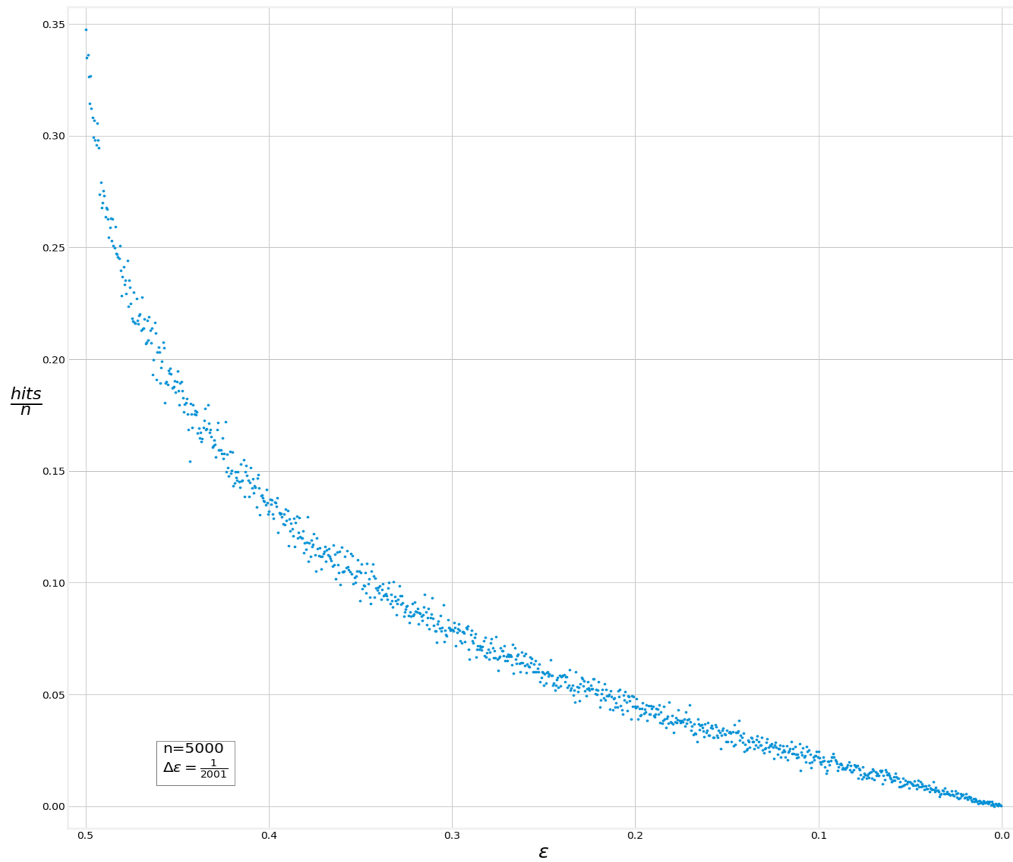

Figure 3.9: A lower bound of the proportion of ε-balanced pairs in $\Delta_4 \times \Delta_4$ as ε changes.

Observation 3.3. Once again this is not the zero function. There are a large number in proportion of these ε-geometrically balanced pairs in $\Delta_4 \times \Delta_4$.

Observation 3.4. When ε is $1/6$ we see at least around $1/20$ of the pairs in $\Delta_4 \times \Delta_4$ are geometrically balanced with the $1/3 - 2/3$ bound.

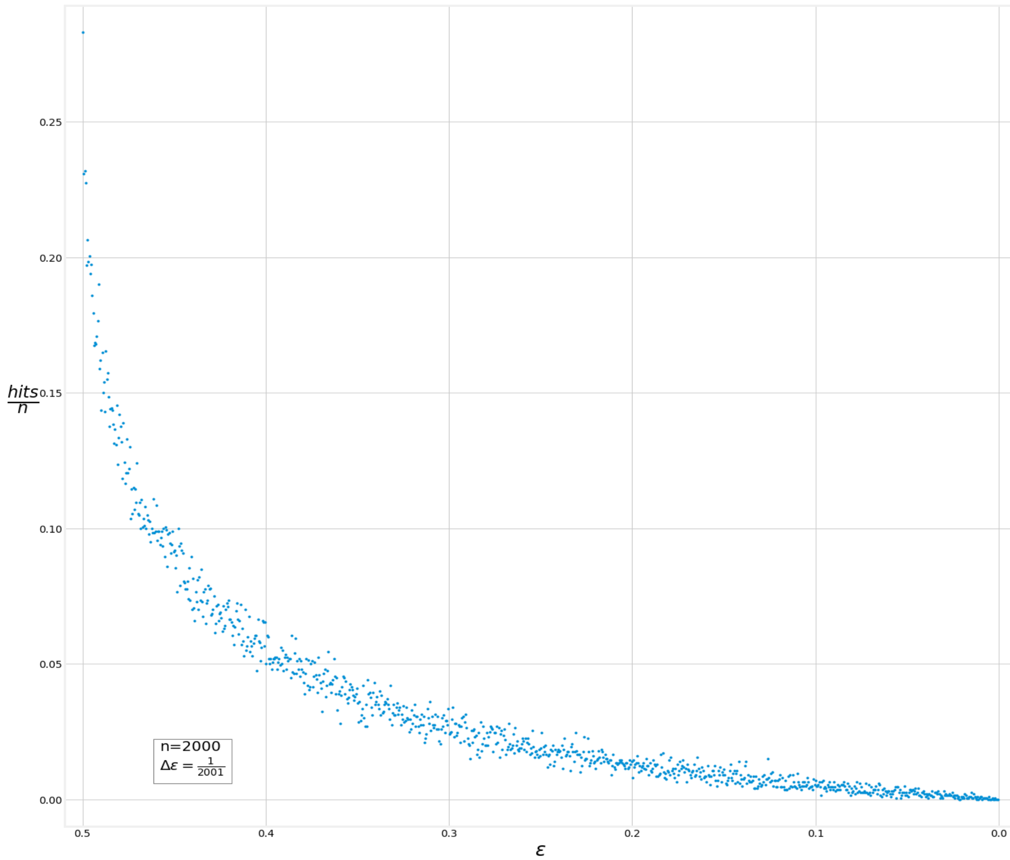

Figure 3.10: A lower bound of the proportion of ε-balanced pairs in $\Delta_5 \times \Delta_5$ as ε changes.

Observation 3.5. Again Figure 3.10. The proportion is nonzero for the number in proportion of these ε-geometrically balanced pairs in $\Delta_5 \times \Delta_5$.

Observation 3.6. When ε is $1/6$ we see at least around $1/100$ of the pairs in $\Delta_5 \times \Delta_5$ are geometrically balanced with the $1/3 - 2/3$ bound.

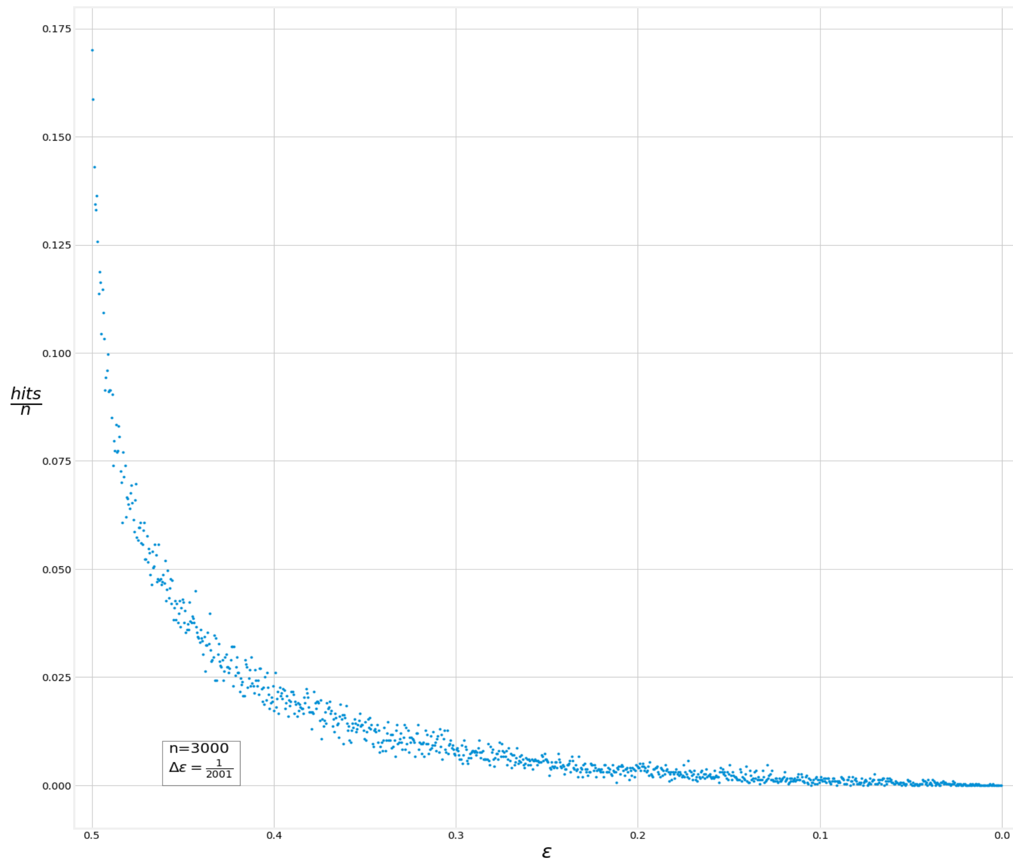

Figure 3.11: A lower bound of the proportion of ε-balanced pairs in $\Delta_6 \times \Delta_6$ as ε changes.

Observation 3.7. We notice in Figures 3.7—3.11 that the general trend becomes more pronounced as dimension increases. We see a steep dropoff as ε gets smaller, but in each case we do have positive evaluations. In other words, even as far as in dimension 6, we observe a nonzero proportion of geometrically ε-balanced pairs even for a very specific pair. This is just a lower bound, because this model only reflects the small (in high dimensions) subset of incomparable pairs that have one difference with a different sign.

Based on our results, we venture to make the following global conjecture.

Conjecture 3.1. *In the poset* $\mathrm{Hom}(C_m, C_n)$*, for any fixed* m*, and as* n *goes the infinity, the proportion of geometrically balanced pairs within all pairs has a positive limit. In other words, the proportion of pairs* x, y *satisfying*

$$\frac{1}{3} \leq \mathbb{P}_{\vartheta}(x < y) \leq \frac{2}{3}$$

is positive.

Appendix I: Python Script

Here is the code script run in python, written with close collaboration with June Overbeck[10]:

```
    %matplotlib inline
import numpy as np
import datetime
import matplotlib.pyplot as plt
from matplotlib import style



def check_epsilon(epsilon, compare_func, n):
    hits = 0
    alpha = [1,1,1,1]
    for i in range(n):
        x, y = np.random.dirichlet(alpha, size=2)
```

```
        z = np.array([y[0]-x[0], (y[0]+y[1])-(x[0]+x[1]),
        (y[0]+y[1]+y[2])-(x[0]+x[1]+x[2])])
        d = find_odd(z)
        if d is not None and 1/2-epsilon < compare_func(z, d)
        < 1/2+epsilon:
            hits = hits + 1
    return hits / n


def find_odd(x):
    c = x
    d = {-1: [], 1: []}
    for j, i in enumerate(np.sign(c)):
        if i in d:
            d[i].append(j)
    if len(d[-1]) != 0 and len(d[1]) != 0:
        return min([d[-1], d[1]], key=lambda x: len(x))[-1]
    else:
        return None
```

```
def comp_func(x, d):
    avail = [0, 1, 2]
    avail.remove(d)
    odd_ball = (x[d])
    reg_1 = (x[avail[-1]]-odd_ball)
    reg_2 = (x[avail[1]]-odd_ball)
    return odd_ball**2/(reg_1*reg_2)


if __name__ == "__main__":
    # Parameters
    steps = 1000
    n = 1000
    f = comp_func

    # Matplotlib stuff
    style.use('fivethirtyeight')
    plt.ion()
    fig, ax = plt.subplots(figsize=(20, 20))
    plot = ax.scatter([], [], s=10)
    ax.set_facecolor("white")
```

```
fig.set_facecolor("white")
plt.xlabel("$\\epsilon$", fontsize=26)
h = plt.ylabel("$\\frac{hits}{n}$", fontsize=32, labelpad=20)
h.set_rotation(0)
box = dict(boxstyle='square, pad=0.4', facecolor='white',
edgecolor='black', alpha=1)
ax.text(.1,.1, "n="+str(n)+"\n$\\Delta\\epsilon=\\frac{1}
{"+str(2*steps+1)+"}$", transform=ax.transAxes, fontsize=20,
verticalalignment='top', bbox=box)
ax.set_xlim((.5, 0))
ax.set_ylim((0, 1))
points = []
for i in [(k+1)/(2*steps+1) for k in range(steps)][::-1]:
    point = np.array([i, check_epsilon(i, f, n)])
    array = plot.get_offsets()
    array = np.append(array, point)
    if array.ndim == 1:
        array = np.reshape(array, (-1,2))
    plot.set_offsets(array)
    fig.canvas.draw()
    ax.set_xlim(array[:, 0].max() + .01, array[:, 0].min()-.01)
```

```
        ax.set_ylim(array[:, 1].min() - .01, array[:, 1].max()+.01)

    fig.savefig(datetime.datetime.now().strftime('%I%MW%m%d%Y')+
    ".png", facecolor=fig.get_facecolor(), edgecolor='none')
    plt.ioff()
plt.show()
```

Here are the changes to go to Δ_4.

```
    def check_epsilon(epsilon, compare_func, n):
    hits = 0
    alpha = [1,1,1,1,1]
    for i in range(n):
        x, y = np.random.dirichlet(alpha, size=2)
        z = np.array([y[0]-x[0], (y[0]+y[1])-(x[0]+x[1]),
        (y[0]+y[1]+y[2])-(x[0]+x[1]+x[2]),(y[0]+y[1]+y[2]+y[3])
        -(x[0]+x[1]+x[2]+x[3])])
        d = find_odd(z)
        if d is not None and 1/2-epsilon < compare_func(z, d)
        < 1/2+epsilon:
            hits = hits + 1
    return hits / n
```

```
def find_odd(x):
    c = x
    d = {-1: [], 1: []}
    for j, i in enumerate(np.sign(c)):
        if i in d:
            d[i].append(j)
    if len(d[-1]) != 0 and len(d[1]) != 0 and len(min([d[-1], d[1]],
    key=lambda x: len(x)))==1:
        return min([d[-1], d[1]], key=lambda x: len(x))[-1]
    else:
        return None


def comp_func(x, d):
    avail = [0, 1, 2,3]
    avail.remove(d)
    odd_ball = (x[d])
    reg_1 = (x[avail[0]]-odd_ball)
    reg_2 = (x[avail[1]]-odd_ball)
    reg_3 = (x[avail[2]]-odd_ball)
```

```
    return abs(odd_ball)**3/abs(reg_1*reg_2*reg_3)
```

Here are the changes to go to Δ_5.

```
    def check_epsilon(epsilon, compare_func, n):
    hits = 0
    alpha = [1,1,1,1,1,1]
    for i in range(n):
        x, y = np.random.dirichlet(alpha, size=2)
        z = np.array([y[0]-x[0], (y[0]+y[1])-(x[0]+x[1]),
        (y[0]+y[1]+y[2])-(x[0]+x[1]+x[2]),(y[0]+y[1]+y[2]+y[3])
        -(x[0]+x[1]+x[2]+x[3]),(y[0]+y[1]+y[2]+y[3]+y[4])-
        (x[0]+x[1]+x[2]+x[3]+x[4])])
        d = find_odd(z)
        if d is not None and 1/2-epsilon < compare_func(z, d)
        < 1/2+epsilon:
            hits = hits + 1
    return hits / n
```

```
def find_odd(x):
```

```
    c = x
    d = {-1: [], 1: []}
    for j, i in enumerate(np.sign(c)):
        if i in d:
            d[i].append(j)
    if len(d[-1]) != 0 and len(d[1]) != 0 and len(min([d[-1], d[1]],
    key=lambda x: len(x)))==1:
        return min([d[-1], d[1]], key=lambda x: len(x))[-1]
    else:
        return None


def comp_func(x, d):
    avail = [0, 1, 2,3,4]
    avail.remove(d)
    odd_ball = (x[d])
    reg_1 = (x[avail[0]]-odd_ball)
    reg_2 = (x[avail[1]]-odd_ball)
    reg_3 = (x[avail[2]]-odd_ball)
    reg_4 = (x[avail[3]]-odd_ball)
    return odd_ball**4/(reg_1*reg_2*reg_3*reg_4)
```

Here are the changes to go to Δ_6.

```
    def check_epsilon(epsilon, compare_func, n):
    hits = 0
    alpha = [1,1,1,1,1,1,1]
    for i in range(n):
        x, y = np.random.dirichlet(alpha, size=2)
        z = np.array([y[0]-x[0], (y[0]+y[1])-(x[0]+x[1]),
        (y[0]+y[1]+y[2])-(x[0]+x[1]+x[2]),(y[0]+y[1]+y[2]+y[3])-
        (x[0]+x[1]+x[2]+x[3]),(y[0]+y[1]+y[2]+y[3]+y[4])
        -(x[0]+x[1]+x[2]+x[3]+x[4]),(y[0]+y[1]+y[2]+y[3]+y[4]+y[5])
        -(x[0]+x[1]+x[2]+x[3]+x[4]+x[5])])
        d = find_odd(z)
        if d is not None and 1/2-epsilon < compare_func(z, d)
        < 1/2+epsilon:
            hits = hits + 1
    return hits / n


def find_odd(x):
    c = x
```

```
    d = {-1: [], 1: []}
    for j, i in enumerate(np.sign(c)):
        if i in d:
            d[i].append(j)
    if len(d[-1]) != 0 and len(d[1]) != 0 and len(min([d[-1], d[1]],
    key=lambda x: len(x)))==1:
        return min([d[-1], d[1]], key=lambda x: len(x))[-1]
    else:
        return None


def comp_func(x, d):
    avail = [0, 1, 2,3,4,5]
    avail.remove(d)
    odd_ball = (x[d])
    reg_1 = (x[avail[0]]-odd_ball)
    reg_2 = (x[avail[1]]-odd_ball)
    reg_3 = (x[avail[2]]-odd_ball)
    reg_4 = (x[avail[3]]-odd_ball)
    reg_5 = (x[avail[4]]-odd_ball)
    return abs(odd_ball)**5/abs(reg_1*reg_2*reg_3*reg_4*reg_5)
```

Appendix II: Python Script Generalized

After the implementations from Appendix I were used, a generalized version of the original code was developed by [10] that has adjustable dimensional constraints and documentation.

```
    import numpy as np
import datetime
import matplotlib.pyplot as plt
from matplotlib import style

""" geobalanced_pairs.py
Monte-Carlo to estimate volume of specific subsets of the order
polytope.
"""

def check_epsilon(epsilon, compare_func, n, dimension):
```

```
""" Estimates the volume of a subset of the order polytope
    Creates a number of random sample points using the dirichlet
    distribution and returns the ratio of points inside the
    subset to the number of samples taken. Uses the parameter
    epsilon to establish the level of geometric balancing
    used to determine inclusion to the desired subset.
    Args:
        epsilon: Defines the interval of a hit
        compare_func: Comparison function used to determine hits
        n: Number of samples to take
        dimension: Dimension of the ambient space
    Returns:
        hit_ratio: The ratio of hits to number of samples
"""
hits = 0
alpha = np.ones(dimension+1)
for i in range(n):
    x, y = np.random.dirichlet(alpha, size=2)
    z = np.array([sum([i for i in x[:k+1]]) -
                  sum([j for j in y[:k]]) for k in
                  range(x.shape[0])])
```

```
        d = find_odd(z)
        if d is not None:
            comp_val = comp_func(z, d)
        if d is not None and 1/2-epsilon < comp_val < 1/2+epsilon:
            hits = hits + 1
    hit_ratio = hits / n
    return hit_ratio


def find_odd(x):
    """ Determines if point with any geometric balancing exists
    Checks a vector for a unique negative or positive element
    Args:
        x: A vector to be examined for a dif
    Returns
        positions: The index with a unique sign in x, or None
                   if there is no unique sign.
    """
    c = x
    d = {-1: [], 1: []}
    for j, i in enumerate(np.sign(c)):
```

```
        if i in d:
            d[i].append(j)
    if len(d[-1]) == 1 or len(d[1]) == 1:
        position = d[-1].pop() if len(d[-1]) == 1 else d[1].pop()
        return position
    else:
        return None


def comp_func(x, d):
    """ Determines the degree of geometric balancing for a particular
    point
    Calculates the level of geometric balancing for a particular point
    Args:
        x: The point of interest
        d: The position where there is a uniquely positive
        or negative element in an array.
    Returns:
        balance: The degree of balancing for x
    """
```

```
    dimension = len(x)
    avail = [i for i in range(dimension)]
    avail.remove(d)
    odd_ball = (x[d])
    denominator = 1
    for i in avail:
        denominator *= (x[i]-odd_ball)
    balance = abs((odd_ball**(dimension-1))/denominator)
    return balance


if __name__ == "__main__":
    # Parameters
    steps = 100
    n = 10000
    dimension = 12

    f = comp_func

    style.use('fivethirtyeight')
    plt.ion()
```

```
fig, ax = plt.subplots(figsize=(20, 20))
plot = ax.scatter([], [], s=10)
ax.set_facecolor("white")
fig.set_facecolor("white")
plt.xlabel("$\\epsilon$", fontsize=26)
h = plt.ylabel("$\\frac{hits}{N}$", fontsize=32, labelpad=20)
h.set_rotation(0)
box = dict(boxstyle='square, pad=0.4', facecolor='white',
           edgecolor='black', alpha=1)
ax.text(.1, .1,
        "N="+str(n)+"\n$\\Delta\\epsilon=\\frac{1}
        {"+str(2*steps+1)+"}$", transform=ax.transAxes,
        fontsize=20, verticalalignment='top',bbox=box)
ax.set_xlim((.5, 0))
ax.set_ylim((0, 1))
points = []

for i in [(k+1)/(2*steps+1) for k in range(steps)][::-1]:
    point = np.array([i, check_epsilon(i, f, n, dimension)])
    ratios = plot.get_offsets()
    ratios = np.append(ratios, point)
```

```
    if ratios.ndim == 1:
        ratios = np.reshape(ratios, (-1, 2))
    plot.set_offsets(ratios)
    fig.canvas.draw()
    ax.set_xlim(ratios[:, 0].max() + .01, ratios[:, 0].min()-.01)
    ax.set_ylim(ratios[:, 1].min() - .01, ratios[:, 1].max()+.01)

fig.savefig(datetime.datetime.now().strftime('%I%MW%m%d%Y')
+".png",
            facecolor=fig.get_facecolor(), edgecolor='none')
plt.ioff()
plt.show()
```

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